Simulating Probabilities, Part 1: Inverse Transform

Simulating Probabilities, Part 2: Monte Carlo

Utility of Money

random.random()

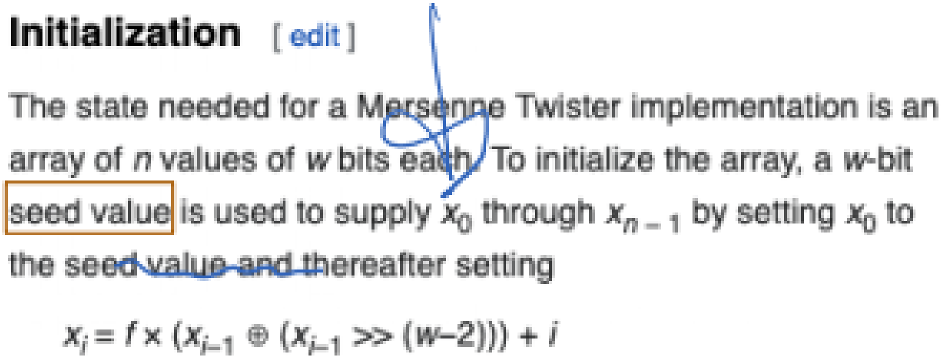
Since computers are deterministic, true randomness does not exist.

We settle for pseudo-randomness: A sequence that looks random but is actually deterministically generated

`random.random()`, `np.random.random()`

- returns a float uniformly in [0.0, 1.0) with the Mersenne Twister:
- 53-bit precision floating point, repeats after 2**19937-1 numbers
- Seed number: X_0 used to generate sequence $X_1, X_2, \ldots, X_n, \ldots$

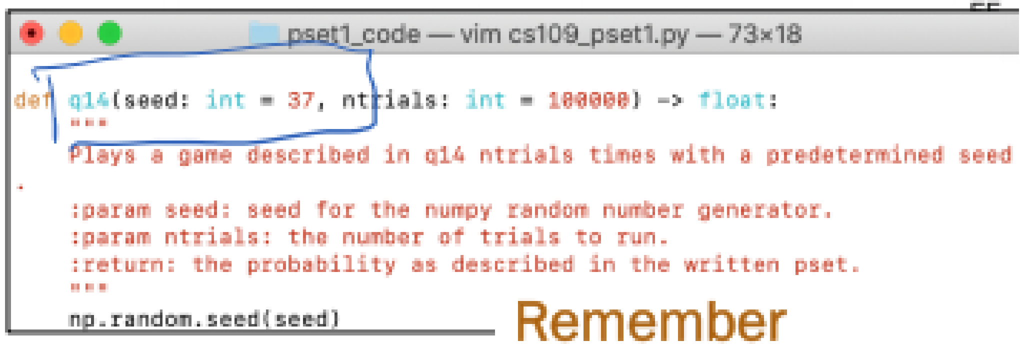
$\frac{1}{2^{53}}$



Initialization [edit]

The state needed for a Mersenne Twister implementation is an array of n values of w bits each. To initialize the array, a w-bit seed value is used to supply x_0 through x_{n-1} by setting x_0 to the seed value and thereafter setting

$$x_i = f \times (x_{i-1} \oplus (x_{i-1} >> (w-2))) + i$$



pset1_code — vim cs109_pset1.py — 73×18

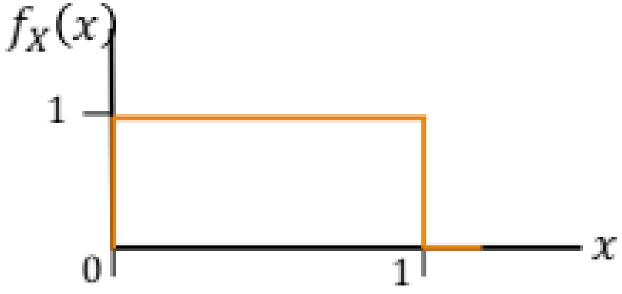
```
def q14(seed: int = 37, ntrials: int = 100000) -> float:
    """
    Plays a game described in q14 ntrials times with a predetermined seed

    :param seed: seed for the numpy random number generator.
    :param ntrials: the number of trials to run.
    :return: the probability as described in the written pset.
    """
    np.random.seed(seed)
```

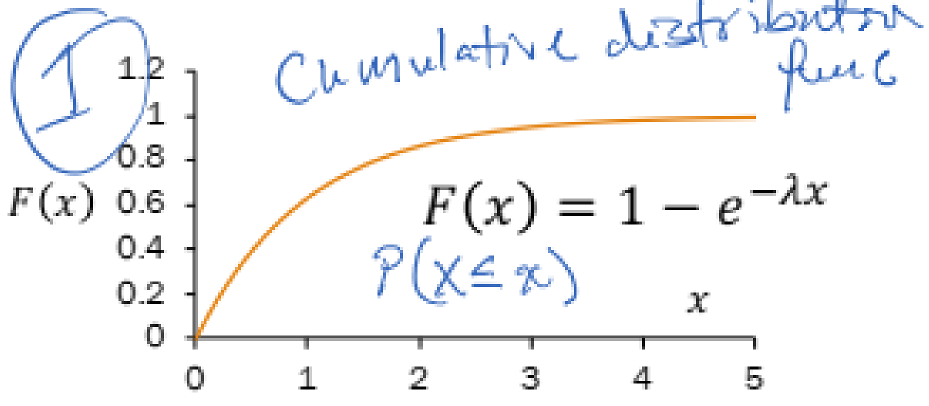
Remember Problem Set 1???

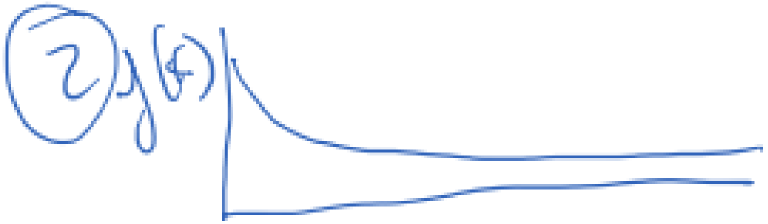
From `random.random()` to everything else

`random.random()`
`np.random.random()`
Generate a random float in interval [0.0, 1.0)
$U \sim \text{Uni}(0,1)$



Generate a random number X according to a distribution
e.g., $X \sim \text{Exp}(\lambda)$





Inverse Transform Sampling

Inverse Transform Sampling

Given the ability to generate numbers $U \sim \text{Uni}(0,1)$, how do we generate another number according to a CDF F?

$$X = F^{-1}(U)$$

<u>def</u> F^{-1} the inverse of CDF: $F^{-1}(a) = b \Leftrightarrow F(b) = a$

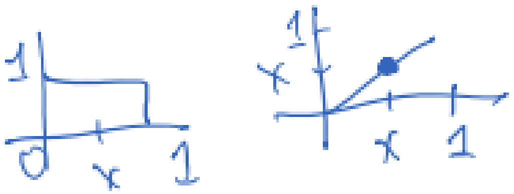
<u>Interpret</u>

1. Generate $U \sim \text{Uni}(0,1)$
2. Apply inverse F^{-1} to get a RV X.
3. Then X will have CDF F.

Proof: $P(X \le x) = P(F^{-1}(U) \le x)$ (our definition of X)

$= P(U \le F(x))$ ($\forall x$: $0 \le F(x) \le 1$)

$= F(x)$ (CDF $P(U \le u) = u$ if $0 \le u \le 1$)



Inverse Transform Sampling (Continuous)

How do we generate the exponential distribution $X \sim \text{Exp}(\lambda)$?

- CDF: $F(x) = 1 - e^{-\lambda x}$ where $x \geq 0$
- Compute inverse:

$$F^{-1}(u) = -\frac{\log(1-u)}{\lambda}$$

- Note if $U \sim \text{Uni}(0,1)$, then $(1-U) \sim \text{Uni}(0,1)$
- Therefore:

$$F^{-1}(U) = -\frac{\log(U)}{\lambda}$$

- Note: Closed-form inverse may not always exist

$F(x) = 1 - e^{-\lambda x} = u$

$1 - u = e^{-\lambda x}$

$\log(1-u) = -\lambda x$

$x = \frac{-\log(1-u)}{\lambda}$

Inverse Transform Sampling (Discrete)

$X \sim \text{Poi}(\lambda = 3)$ has CDF $F(X = x)$ as shown:

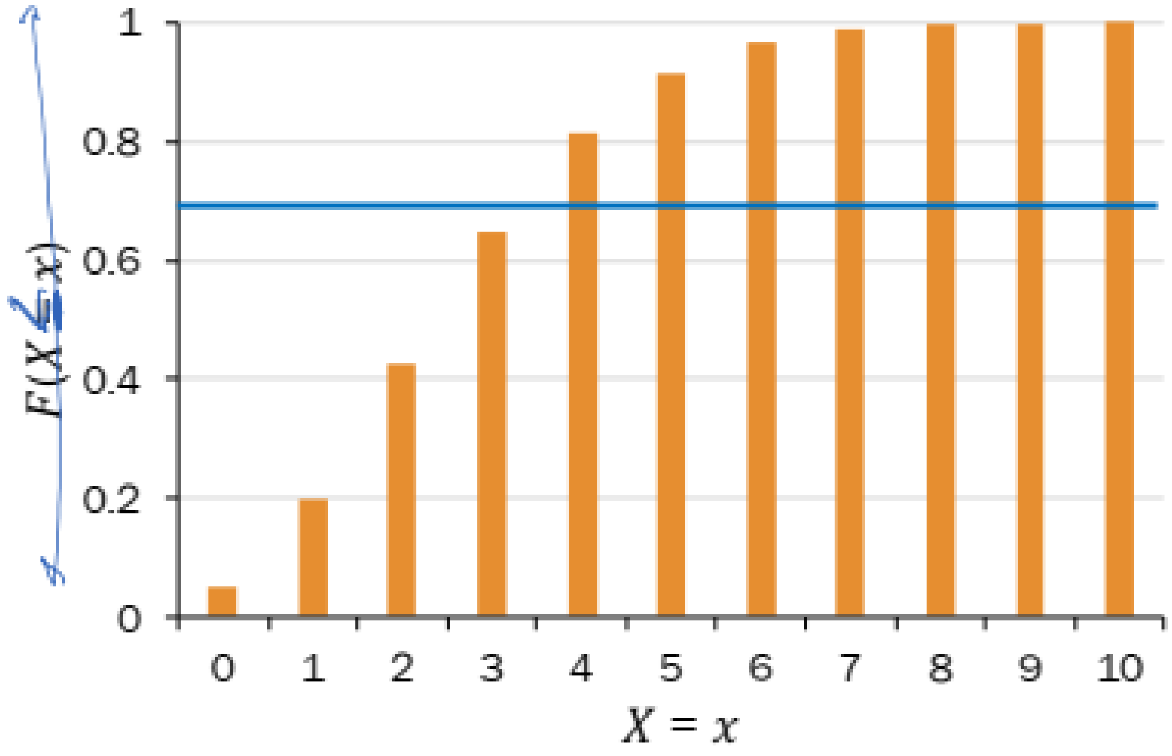
1. Generate $U \sim \text{Uni}(0,1)$

   $u = 0.7$

2. As x increases, determine first $F(x) \geq U$

   $x = 4$

3. Return this value of x



Inverse Transform Sampling of the Normal?

How do we generate $X \sim \mathcal{N}(0,1)$?

Inverse transform sampling:

1. Generate a random probability u from $U \sim \text{Unif}(0,1)$.
2. Find x such that $\Phi(x) = u$. In other words, compute $x = \Phi^{-1}(u)$.

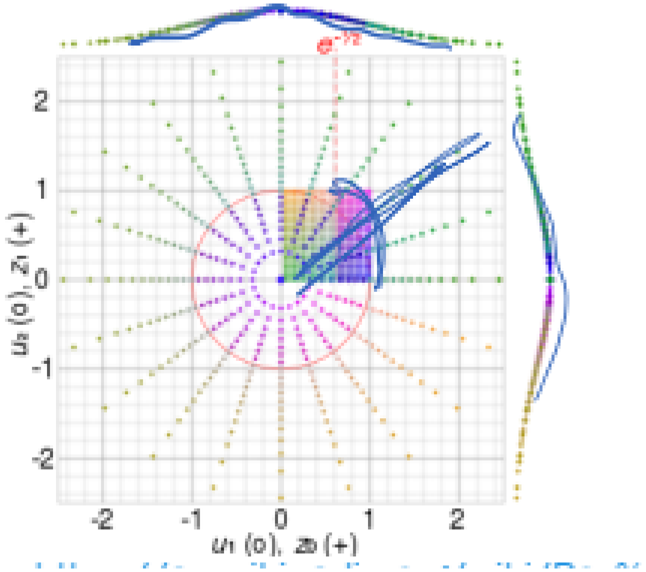
⚠ Φ^{-1} has no analytical solution!

<u>Solution</u> Box-Muller Transform

- Use **two** uniforms U_1 and U_2 to generate polar coordinates R and Θ for a circle inscribed in 2x2 square centered at (0,0)
- Can define $X = R\cos\Theta, Y = R\sin\Theta$ such that X and Y are **two** independent unit Normals

$R = \sqrt{-2\log U_1}$

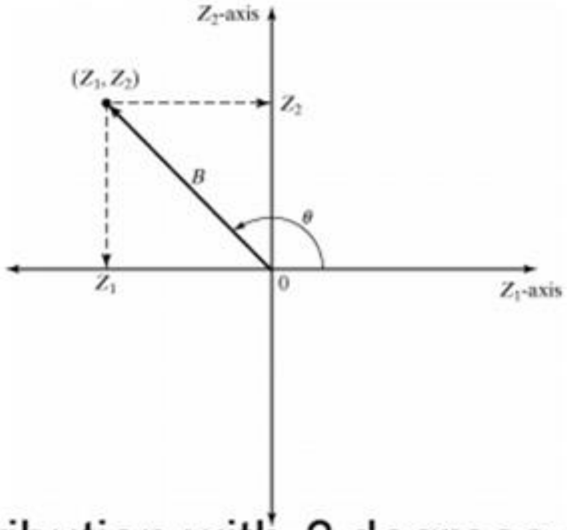
$\Theta = 2\pi U_2$



Direct Transformation

[Special Properties]

- Approach for normal*(0,1)*:
  - Consider two standard normal random variables, Z_1 and Z_2, plotted as a point in the plane:



In polar coordinates:

$$Z_1 = B \cos \phi$$
$$Z_2 = B \sin \phi$$

  - $B^2 = Z^2_1 + Z^2_2$ ~ chi-square distribution with 2 degrees of freedom = $Exp(\lambda = 2)$. Hence, $B = (-2 \ln u_1)^{1/2}$
  - The radius B and angle ϕ are mutually independent.

$$Z_1 = (-2 \ln u_1)^{1/2} \cos(2\pi u_2)$$
$$Z_2 = (-2 \ln u_1)^{1/2} \sin(2\pi u_2)$$

Direct Transformation

- Approach for normal(μ, σ^2):
  - Generate $Z_i \sim N(0,1)$

$$X_i = \mu + \sigma Z_i$$

- Approach for lognormal(μ, σ^2):
  - Generate $X \sim N((\mu, \sigma^2)$

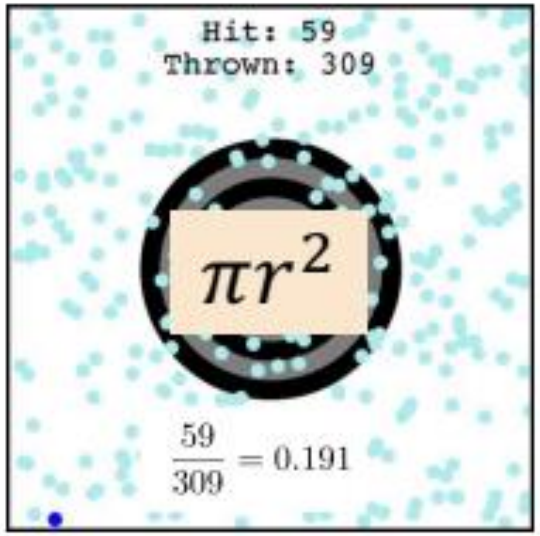
$$Y_i = e^{X_i}$$

Monte Carlo Methods

Monte Carlo Integration

Monte Carlo methods: randomly sample repeatedly to obtain a numerical result

- Bootstrap
- Inference in Bayes Nets
- Definite integrals (Monte Carlo integration)

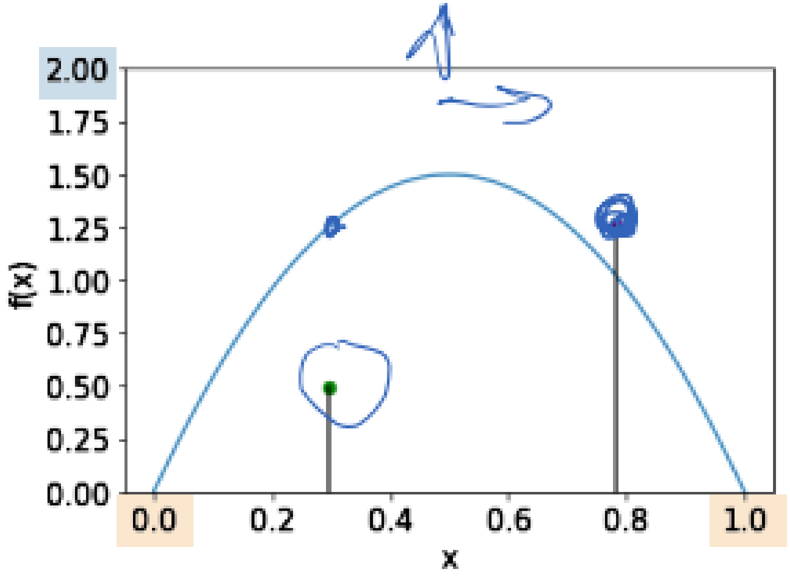


A Monte Carlo method: Rejection Filtering

Lisa would rename to Acceptance Filtering

Idea for X with PDF $f(x)$:

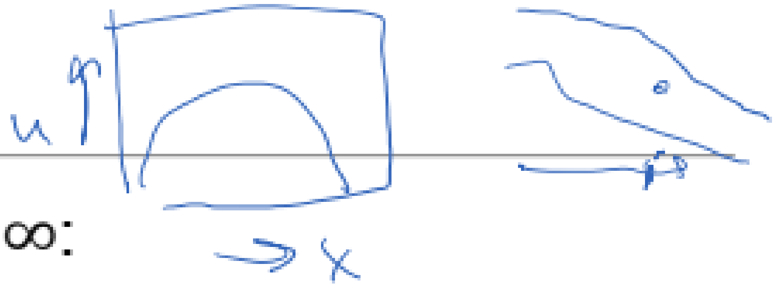
- Throw dart at graph of PDF $f(x)$
- If dart under $f(x)$: return x
- Otherwise, repeat throwing darts until one lands under $f(x)$




```
# random value from distr of X
def random_x():
  while True:
    u = random.random() * HEIGHT
    x = random.random() * WIDTH
    if u <= f(x):
      return x
```

But what if our PDF has infinite support?

Filtering with infinite support



Idea for X with PDF $f(x)$ with support $-\infty < x < \infty$:

- Suppose we can simulate Y with PDF $g(y)$ (where Y has same support as X)
- If we can find a constant c such that $c \geq f(x)/g(x)$ for all x, then

```
def random_x():
  while True:
    u = random.random()  # u ~ Uni(0, 1)
    x = generate_y()     # random value Y = y
    if u <= f(x)/(c * g(x)):
      return x
```

- Number of iterations of loop~Geo$(1/c)$
- Proof of correctness in Ross textbook, 10.2.2

Generating Normal Random Variable

$g(y) = e^{-y}$

$0 \leq y < \infty$

Goal: Simulate $Z \sim \mathcal{N}(0, 1)$.

- Suppose we can simulate $Y \sim \text{Exp}(1)$ with the inverse transform.
- Let's simulate $X = |Z|$, which has the same support as Y. PDF f:

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}$$

$0 \leq x < \infty$

1. Determine constant $c \geq f(x)/g(x)$ for all $0 \leq x < \infty$:

$$\frac{f(x)}{g(x)} = \sqrt{\frac{2}{\pi}} e^{-(x^2-2x)/2} = \sqrt{\frac{2}{\pi}} e^{-(x^2-2x+1)/2 + 1/2} = \sqrt{\frac{2e}{\pi}} e^{-(x-1)^2/2} \leq \sqrt{\frac{2e}{\pi}}$$

(complete the square) $\quad (e^{1/2} = \sqrt{e})$

Let this be c

2. Determine $f(x)/(cg(x))$

3. Implement code for $|Z|$ and Z

Generating Normal Random Variable

Goal: Simulate $Z \sim \mathcal{N}(0,1)$.

$g(y) = e^{-y}$
$0 \le y < \infty$

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$0 \le x < \infty$

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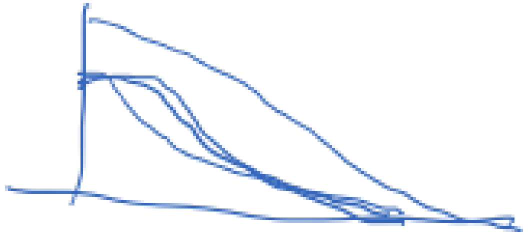
$$\frac{f(x)}{g(x)} = \sqrt{\frac{2}{\pi}} e^{-(x^2-2x)/2} = \sqrt{\frac{2}{\pi}} e^{-(x^2-2x+1)/2+1/2} = \sqrt{\frac{2e}{\pi}} e^{-(x-1)^2/2} \le \sqrt{\frac{2e}{\pi}}$$

(complete the square) ($e^{1/2} = \sqrt{e}$)

Let this be c

2. Determine $f(x)/(c \cdot g(x))$

$$e^{-(x-1)^2/2}$$



3. Implement code for $|Z|$ and Z

Generating Normal Random Variable

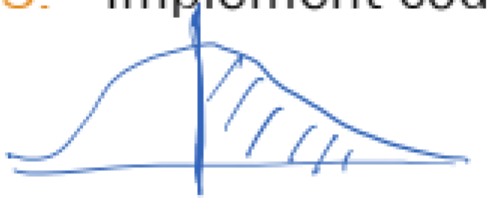
Goal: Simulate $Z \sim \mathcal{N}(0, 1)$.

- Suppose we can simulate $Y \sim \text{Exp}(1)$ with the inverse transform.
- Let's simulate $X = |Z|$, which has the same support as Y. PDF f:

$$g(y) = e^{-y} \quad 0 \le y < \infty$$

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2} \quad 0 \le x < \infty$$

3. Implement code for $|Z|$ and Z.



$$\frac{f(x)}{c \cdot g(x)} = e^{-(x-1)^2/2}$$

$$c = \sqrt{2e/\pi} \approx 1.32$$

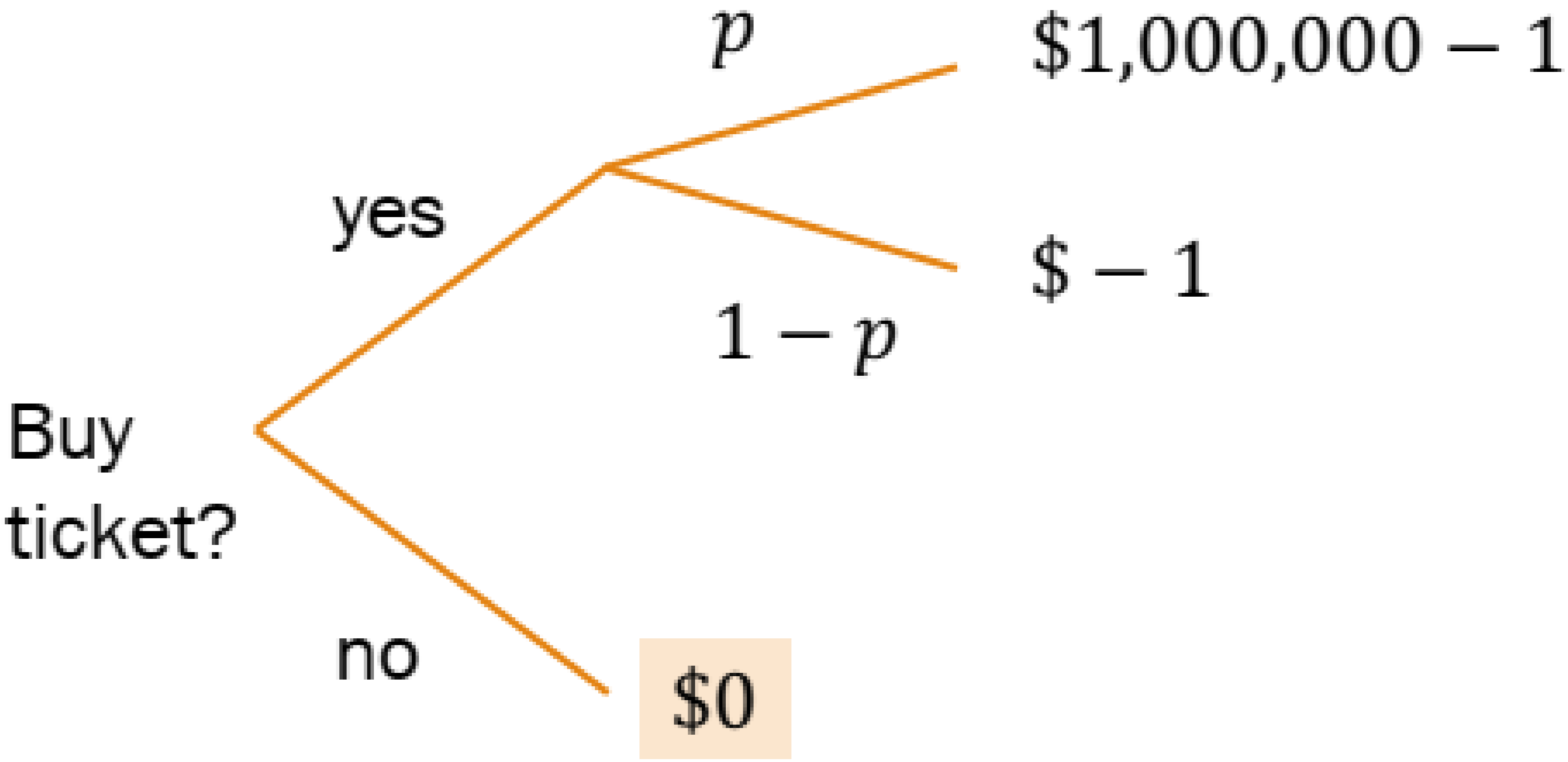
(from last two slides)

```python
# random value from distr of |Z|
def random_abs_z():
  while True:
    u = random.random()  # u ~ Uni(0, 1)
    # inverse transform to get x ~ Exp(1)
    x = -np.log(random.random())
    if u <= np.exp(-(x - 1) ** 2 / 2):
      return x
```

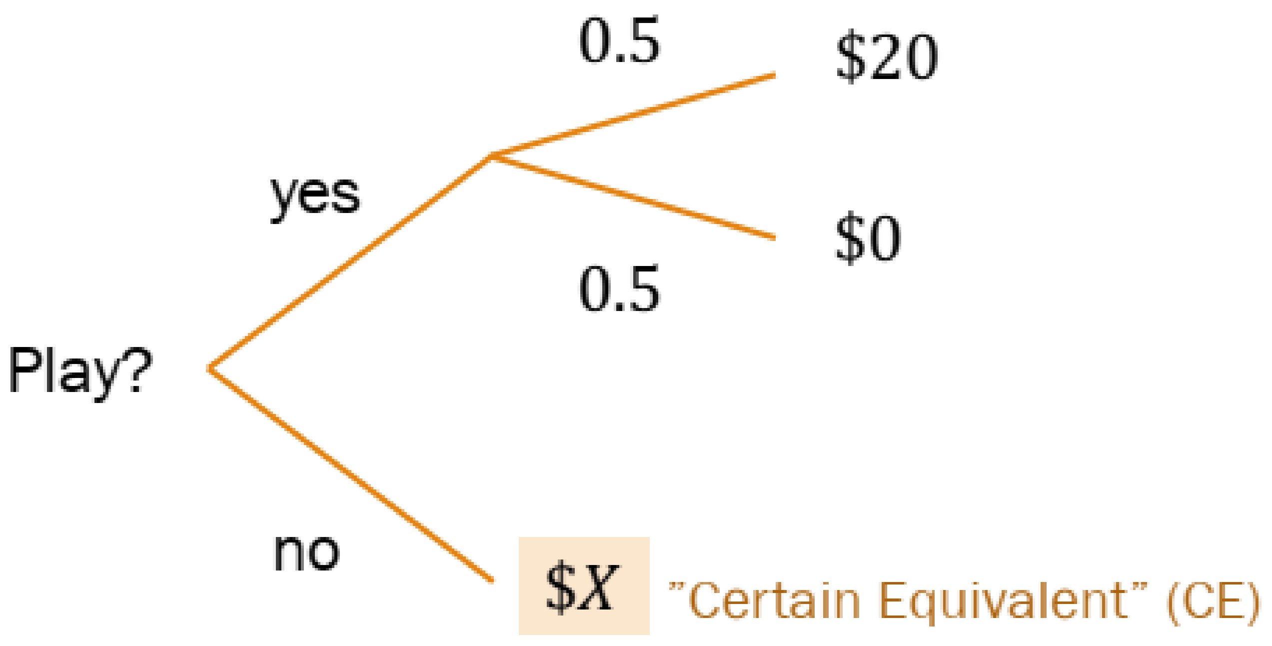
```python
# random value from distr of Z
def random_z():
  abs_z = random_abs_z()
  u = random.random()
  if u < 0.5:
    return abs_z
  else:
    return -abs_z
```

Utility of Money

Recall the probability tree!



Let's play a game. What choice would you make?



For what value of $\$X$ are you <u>indifferent</u> to playing?

A. $X = 3$
B. $X = 7$
C. $X = 9$
D. $X = 10$

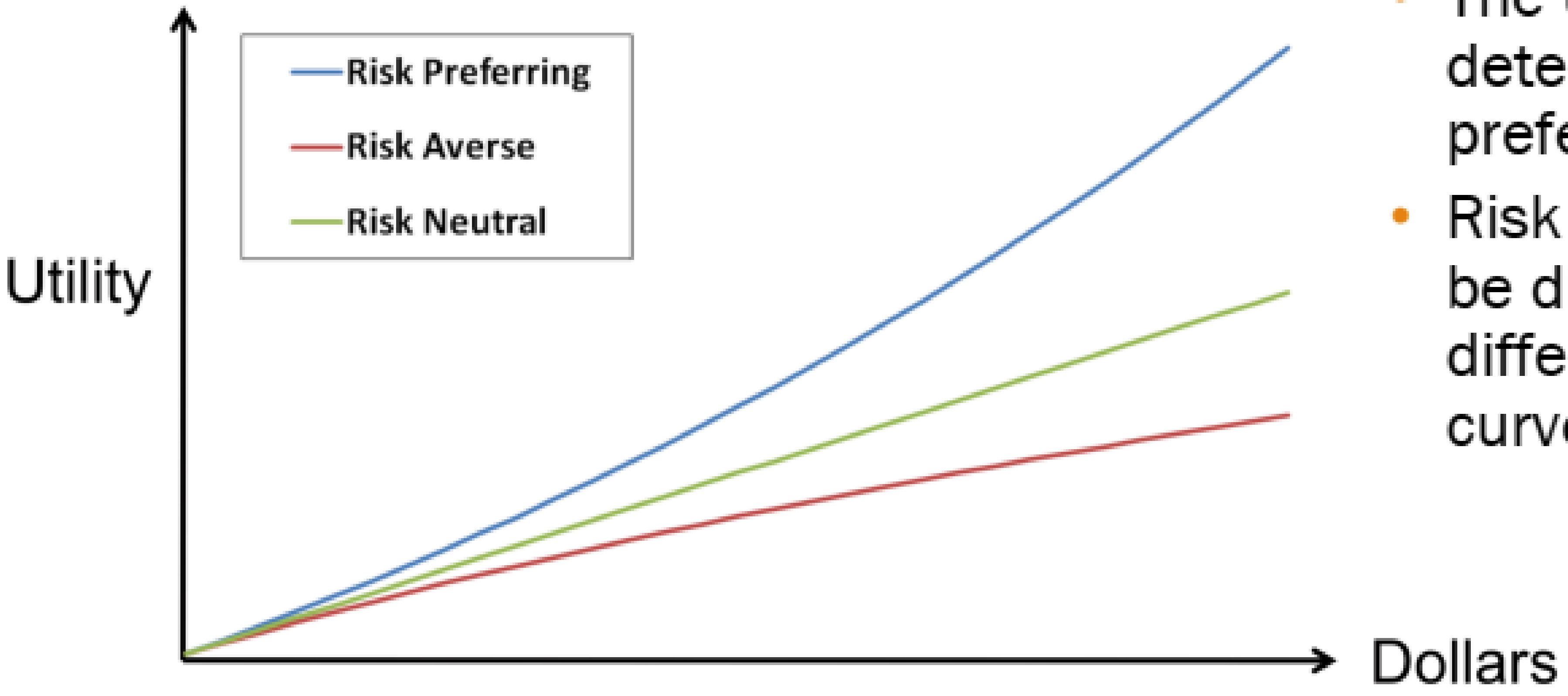
<u>def</u> Certain equivalent: The value of the game to *you* (different for different people)

Utility

<u>def</u> Utility $U(X)$ is the "value" you derive from X

- Can be monetary, but often includes intangibles like quality of life, life expectancy, personal beliefs, etc.

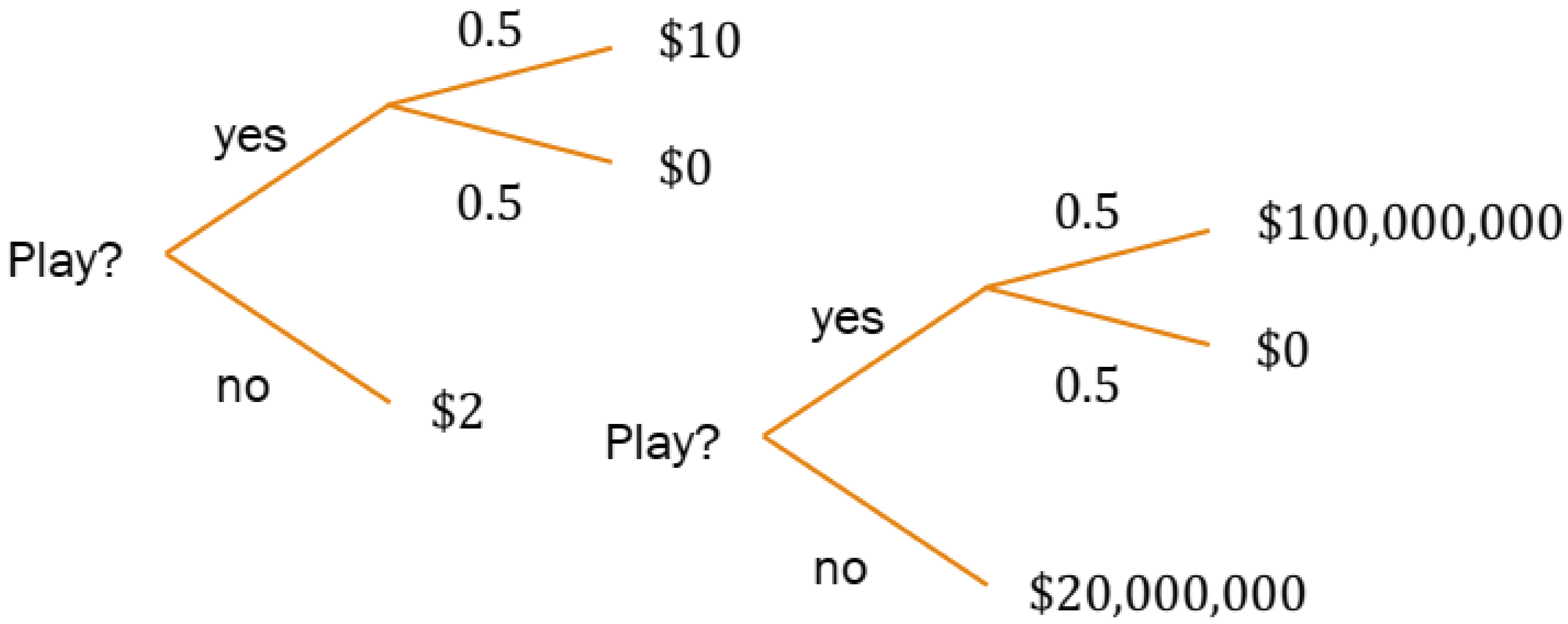
Utility curves



- The utility curve determines your "risk preference."
- Risk preference can be different in different parts of the curve

Non-linearity utility of money

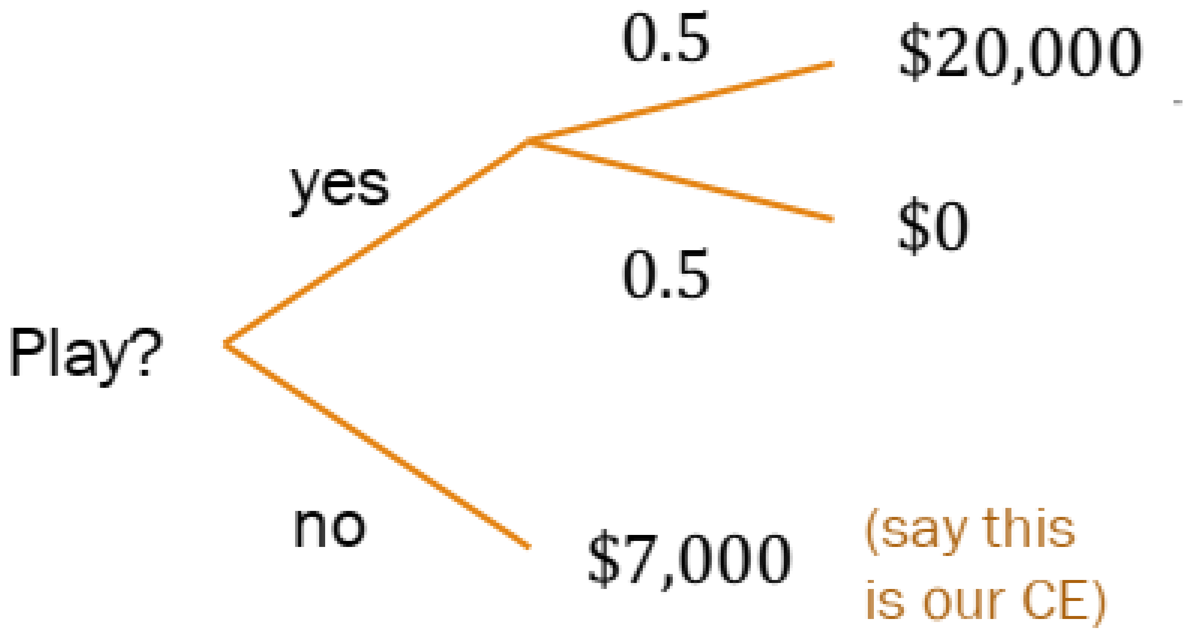
Interestingly, these two choices are different for most people:



Insurance and risk premium

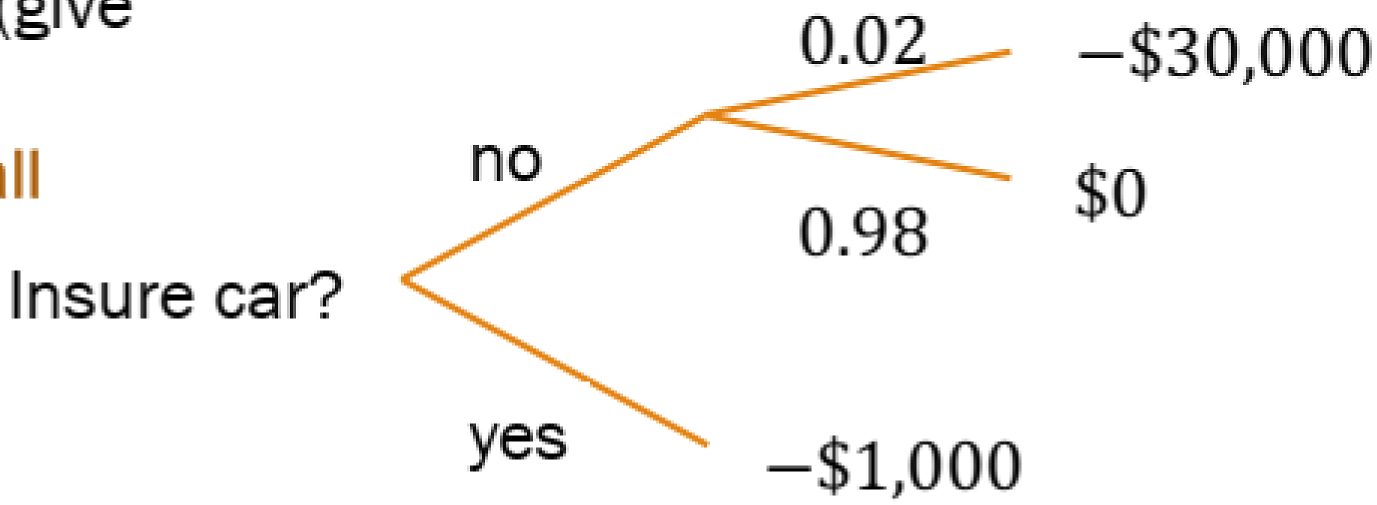
A slightly different game:

- Expected monetary value (EMV) = expected dollar value of game (here, $10,000)



Risk premium = EMV – CE = $3000

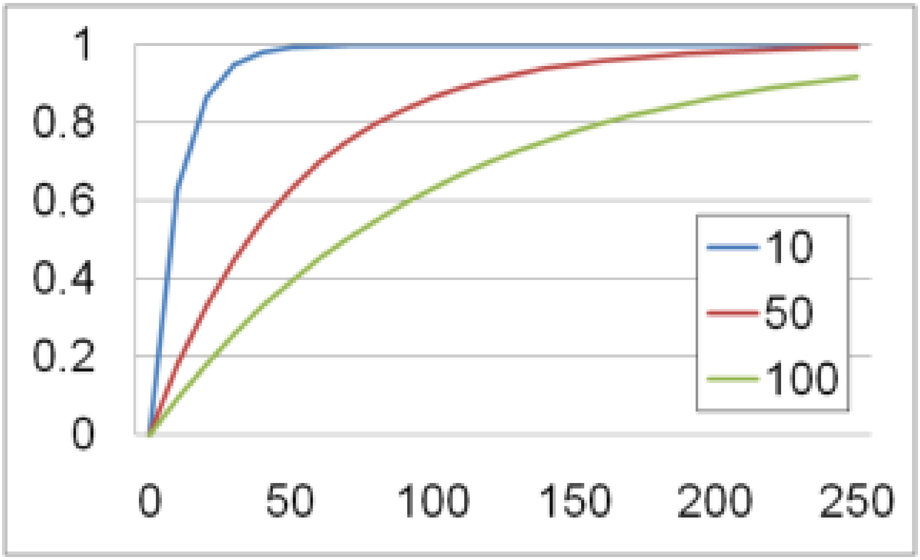
- How much would you pay (give up) to avoid risk?
- This is what insurance is all about.



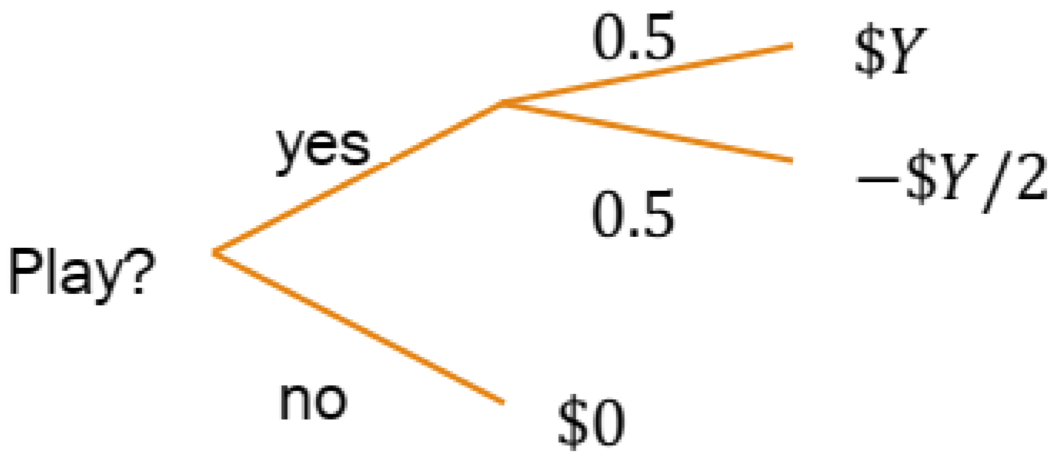
Exponential utility curves

Many people have exponential utility curves:

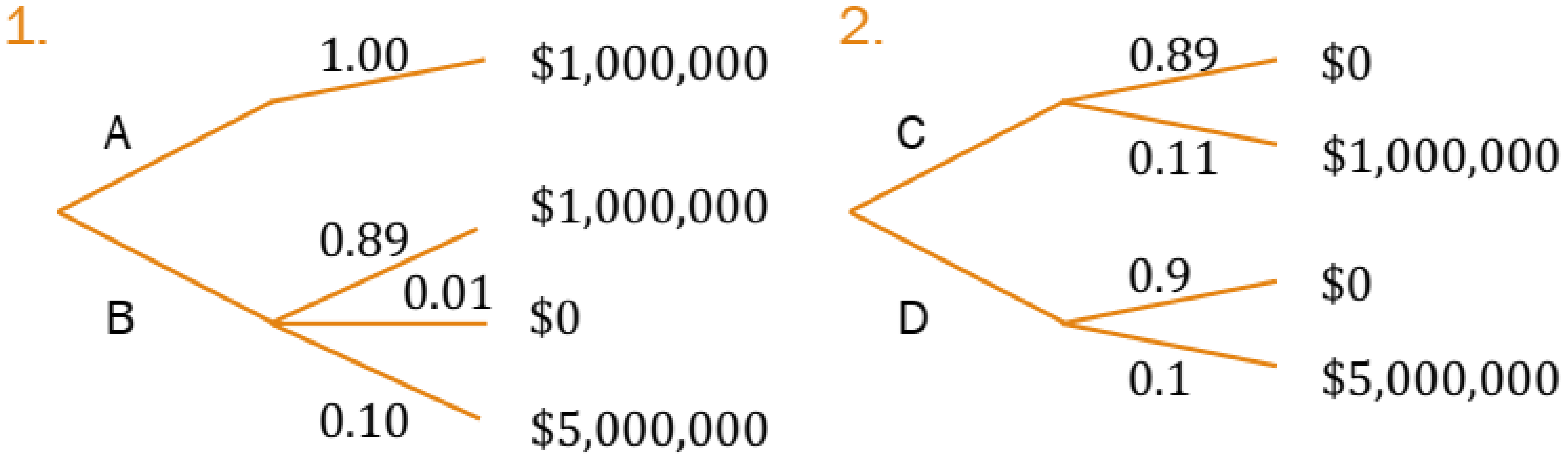
$$U(x) = 1 - e^{-x/R}$$



- R is your "risk tolerance"
- Larger R = less risk aversion. Makes utility function more "linear"
- $R \approx$ highest value of Y for which you would play:



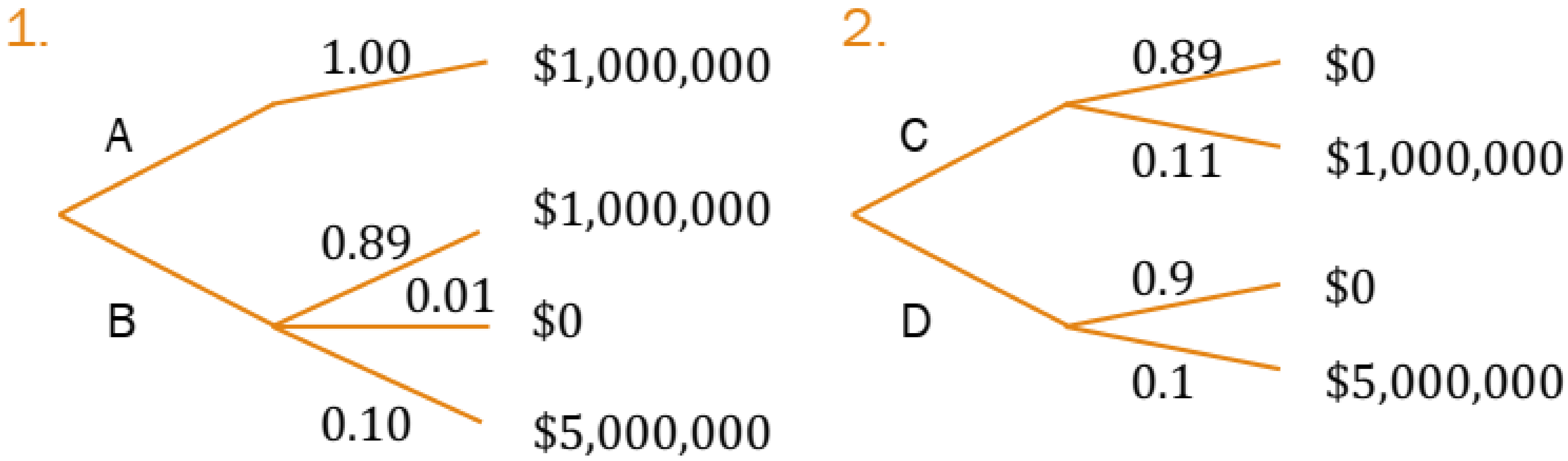
How rational are you?



Which option would you choose in each case?

How many of you chose A over B and D over C?

How rational are you?



Choice A preferred:
$1.00\,U(1{,}000{,}000) > 0.89\,U(1{,}000{,}000) + 0.01\,U(0) + 0.10\,U(5{,}000{,}000)$

Choice D preferred:
$0.89\,U(0) + 0.11\,U(1{,}000{,}000) < 0.90\,U(0) + 0.10\,U(5{,}000{,}000)$

How rational are you?

Choice D preferred:
$0.11\ U(1{,}000{,}000) < 0.01\ U(0) + 0.10\ U(5{,}000{,}000)$

add $0.89\ U(1{,}000{,}000)$ to both sides

Choice D preferred:
$1.00\ U(1{,}000{,}000) < 0.89\ U(1{,}000{,}000) + 0.01\ U(0) + 0.10\ U(5{,}000{,}000)$

Contradiction???

Choice A preferred:
$1.00\ U(1{,}000{,}000) > 0.89\ U(1{,}000{,}000) + 0.01\ U(0) + 0.10\ U(5{,}000{,}000)$

subtract $0.89\ U(0)$ from both sides

Choice D preferred:
$0.89\ U(0) + 0.11\ U(1{,}000{,}000) < 0.90\ U(0) + 0.10\ U(5{,}000{,}000)$

How rational are you?

Choice D preferred:
$1.00\ U(1{,}000{,}000) <$
$0.89\ U(1{,}000{,}000) +$
$0.01\ U(0) +$
$0.10\ U(5{,}000{,}000)$

add
$0.89\ U(1{,}000{,}000)$
to both sid

Choice D preferred:
$0.11\ U(1{,}00$

Con

subtract $0.89\ U(0)$
from both sides

Cho
1.00
$,000{,}000) >$
$0.89\ U(1{,}000{,}000) + 0.01\ U(0)$
$+0.10\ U(5{,}000{,}000)$

Choice D preferred:
$0.89\ U(0) + 0.11\ U(1{,}000{,}000) <$
$0.90\ U(0) + 0.10\ U(5{,}000{,}000)$

You are inconsistent with utility theory (Allais Paradox)!
Human behavior is not always axiomatically consistent